Josephson Effect: Physics and Applications

Sergio Pagano

Dipartimento di Matematica e Informatica

Università di Salerno

Italy

Outline of the lessons:

- **Basics of Josephson effect**
- **Fabrication Technology**
- **Applications**

Basics of Josephson effect:

- Tunneling among superconductors
- Josephson Equations
- DC and AC Josephson effect
- Microscopic Theory
- Electrodynamics of a Josephson junction
- Magnetic field effects
- RF fields effects
- Nonlinear waves in Josephson junctions

A superconductor can be seen as a macroscopic quantum state

$$\psi = \rho^{1/2} e^{j\varphi}$$

where φ is the phase common to all the particles and ρ represents, in this macroscopic picture, their actual density in the macrostate |s> :

$$\langle s|\psi^* \psi|s\rangle = |\psi|^2 = \rho$$

The electric current density can be written, in the presence of a vector potential A:

$$\mathbf{J} = \frac{e^*}{m^*}\left[\frac{j\hbar}{2}(\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{e^*}{c}\mathbf{A}|\psi|^2\right]$$

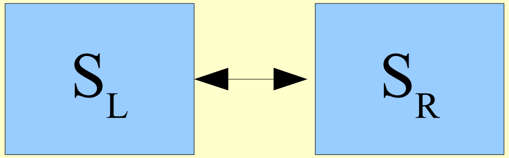
or

$$\mathbf{J} = \rho \frac{e}{m}\left(\hbar \nabla \varphi - \frac{2e}{c}\mathbf{A}\right)$$

The time evolution of ψ in stationary conditions obeys the usual quantum mechanical equation of the form:

$$j\hbar \frac{\partial \psi}{\partial t} = E\psi$$

Coupled Superconductors



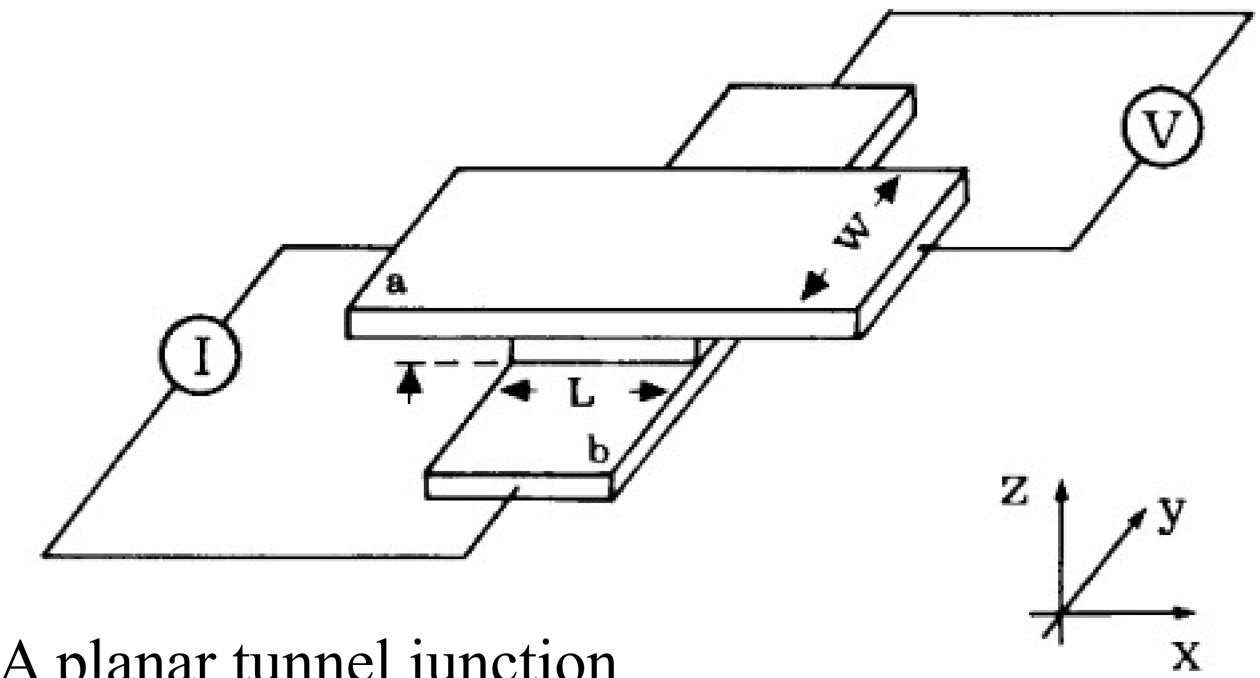
Let us now consider two superconductors S_L and S_R separated by a macroscopic distance. In this situation, the phase of the two superconductors can change independently.

As the two superconductors are moved closer, so that their separation is reduced to about 3 nm, quasiparticles can flow from one superconductor to the other by means of tunneling (single electron tunneling).

If we reduce further the distance between S_L and S_R down to say 1 nm, then also Cooper pairs can flow from one superconductor to the other (Josephson tunneling). The long range order is "transmitted" across the boundary.

The whole system of the two superconductors separated by a thin dielectric barrier will behave as a single superconductor. This phenomenon is often called "weak superconductivity" (Anderson 1963) because of the much lower values of the critical parameters involved.

Quasiparticle tunneling



A planar tunnel junction

$$I_{L\to R}=\frac{2\pi}{\hbar}\int_{-\infty}^{+\infty}|\mathrm{T}|^2 N_L(E)f_L(E)N_R(E)(1-f_R(E))\,dE$$

$$I=I_{L\to R}-I_{R\to L}=\frac{2\pi}{\hbar}\int_{-\infty}^{+\infty}|\mathrm{T}|^2 N_L(E)N_R(E)[f_L(E)-f_R(E)]\,dE$$

$$I=\frac{2\pi}{\hbar}|\mathrm{T}|^2\int_{-\infty}^{+\infty}N_L(E)N_R(E+eV)[f_L(E)-f_R(E+eV)]\,dE$$

Let us consider two normal metals. Assume also N_L and N_R to be constant and equal to the densities of states at the Fermi energy level.

$$I_{NN} = \text{constant} \times \int_{-\infty}^{+\infty} [f(E) - f(E+eV)]\, dE$$

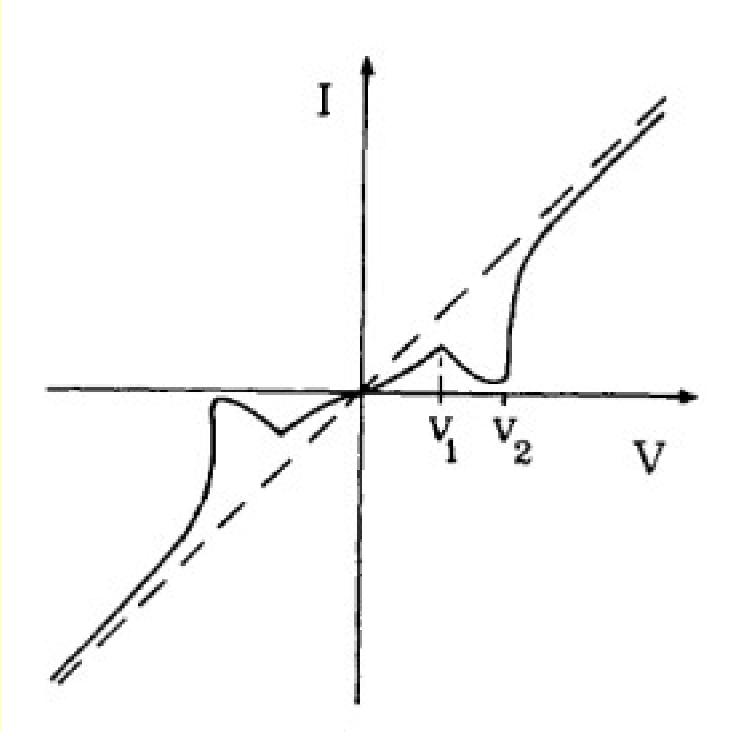
that is

$$I_{NN} = \sigma_N V$$

When the two metals are in the superconducting state the situation is greatly altered. In fact the densities of states are now given by:

$$N(E) = N(0) \frac{E}{\sqrt{E^2 - \Delta^2}} \qquad |E| \geqslant \Delta$$

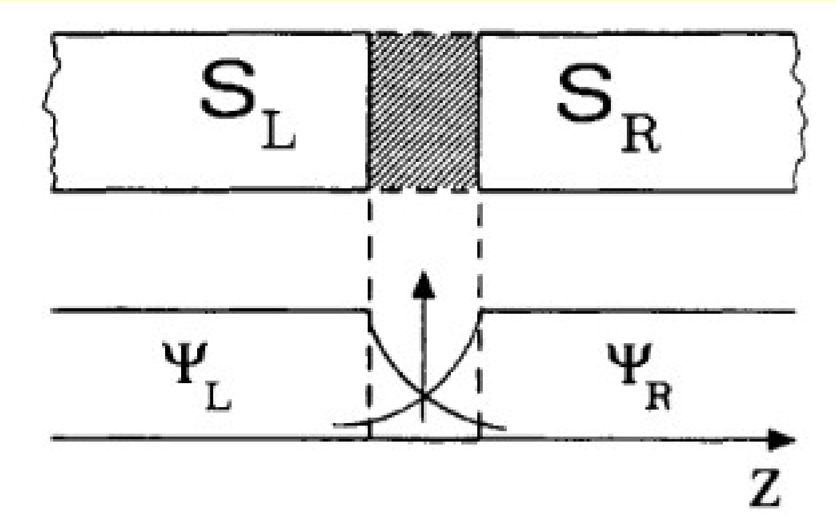
$$N(E) = 0 \qquad |E| < \Delta$$



Therefore the tunneling current in a junction with the electrodes both superconductors is given by

$$I_{SS} = \text{constant} \times \int_{-\infty}^{+\infty} \frac{|E|}{|E^2 - \Delta_L^2|^{1/2}} \frac{|E+eV|}{|(E+eV)^2 - \Delta_R^2|^{1/2}} [f(E) - f(E+eV)]\, dE$$

Cooper pair tunneling



simple derivation due to Feynman based on a weakly coupled "two level system" picture.

$$\langle L|\psi_L^*\psi_L|L\rangle = |\psi_L|^2 = \rho_L \qquad \langle R|\psi_R^*\psi_R|R\rangle = |\psi_R|^2 = \rho_R$$

$$|\psi\rangle = \psi_R|R\rangle + \psi_L|L\rangle \qquad j\hbar\frac{\partial|\psi\rangle}{\partial t} = \mathcal{H}|\psi\rangle$$

$$\mathcal{H} = \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_T \qquad \mathcal{H}_L = E_L|L\rangle\langle L| \qquad \mathcal{H}_R = E_R|R\rangle\langle R| \qquad \mathcal{H}_T = K\left[|L\rangle\langle R| + |R\rangle\langle L|\right]$$

$$j\hbar\frac{\partial\psi_R}{\partial t} = E_R\psi_R + K\psi_L$$

$$j\hbar\frac{\partial\psi_L}{\partial t} = E_L\psi_L + K\psi_R$$

E_L and E_R are the ground state energies and K is the coupling amplitude. If we consider a d.c. voltage V across the junction the ground states are shifted by an amount eV and consequently it is E_L-E_R = 2eV.

$$j\hbar\frac{\partial\psi_R}{\partial t} = -eV\psi_R + K\psi_L$$

$$j\hbar\frac{\partial\psi_L}{\partial t} = eV\psi_L + K\psi_R$$

$$\psi_L = \rho_L^{1/2}e^{j\varphi_L} \qquad \psi_R = \rho_R^{1/2}e^{j\varphi_R}$$

$$\psi_L = \rho_L^{1/2} e^{j\varphi_L} \qquad \psi_R = \rho_R^{1/2} e^{j\varphi_R}$$

$$\frac{\partial \rho_L}{\partial t} = \frac{2}{\hbar} K \sqrt{\rho_L \rho_R} \sin\varphi$$

$$\frac{\partial \rho_R}{\partial t} = -\frac{2}{\hbar} K \sqrt{\rho_L \rho_R} \sin\varphi$$

$$\frac{\partial \varphi_L}{\partial t} = \frac{K}{\hbar} \sqrt{\frac{\rho_L}{\rho_R}} \cos\varphi + \frac{eV}{\hbar}$$

$$\frac{\partial \varphi_R}{\partial t} = \frac{K}{\hbar} \sqrt{\frac{\rho_L}{\rho_R}} \cos\varphi - \frac{eV}{\hbar}$$

$$\varphi = \varphi_L - \varphi_R$$

$$J \equiv \frac{\partial \rho_L}{\partial t} = -\frac{\partial \rho_R}{\partial t}$$

$$J = \frac{2K}{\hbar} \sqrt{\rho_L \rho_R} \sin\varphi$$

$$\frac{\partial \varphi}{\partial t} = \frac{2eV}{\hbar}$$

The Josephson equations

Assuming ρ_L and ρ_R constant (the current source continuously replaces the pairs tunneling across the barrier.

$$J = J_1 \sin\varphi$$

$$J = J_1 \sin\varphi \qquad \frac{\partial\varphi}{\partial t} = \frac{2eV}{\hbar}$$

Assuming V=0 the phase difference φ results to be constant not necessarily zero, so that a finite current density with a maximum value J_1 can flow through the barrier with zero voltage drop across the junction.
This is the essence of the **d.c. Josephson effect** (Josephson 1962). The first observation was made by Anderson and Rowell in 1963.

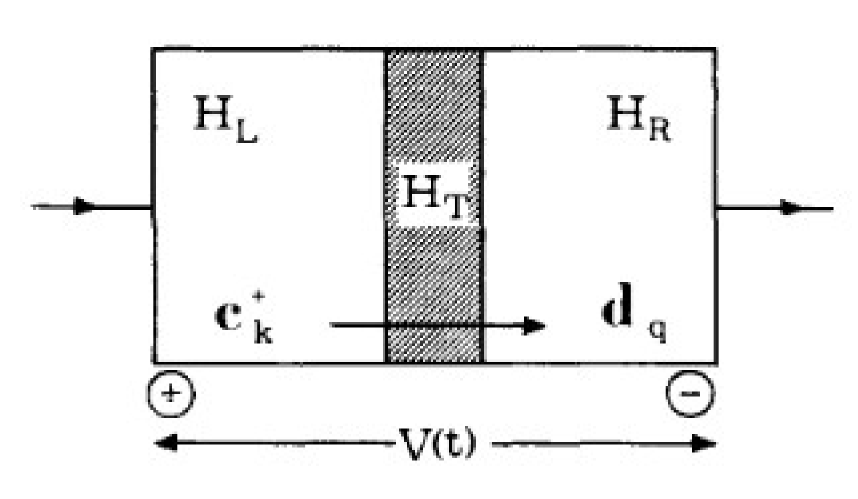
If we apply a constant voltage V=0, it follows that the phase φ varies in time as $\varphi = \varphi_0 + 2eV/\hbar\, t$ and therefore there appears an alternating current

$$J = J_1 \sin\left(\varphi_0 + \frac{2e}{\hbar} V t\right)$$

with a frequency ω = 2πf = 2eV/ħ. This is called **a.c. Josephson effect**.
The ratio between frequency and voltage is:

f/V = 483.6 MHz/μV

Microscopic theory of Josephson effect: Tunneling Hamiltonian formalism



$$\mathcal{H} = \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_T$$

$$N_L = \sum_{\mathbf{k},\sigma} c^+_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}; \qquad N_L = \sum_{\mathbf{q},\sigma} d^+_{\mathbf{q}\sigma} d_{\mathbf{q}\sigma}$$

$$\mathcal{H}_T = \sum_{\mathbf{kq}\sigma} \left[\mathrm{T}_{\mathbf{kq}} c^+_{\mathbf{k}\sigma} d_{\mathbf{q}\sigma} + \mathrm{T}^*_{\mathbf{kq}} d^+_{\mathbf{q}\sigma} c_{\mathbf{k}\sigma} \right]$$

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} V(t)$$

$$I(V,T) = -e\langle \dot{N}_R \rangle$$

$$I(t,V_0,T) = I_{qp}(V_0,T) + I_{J1}(V_0,T)\sin\varphi(t) + I_{J2}(V_0,T)\cos\varphi(t)$$

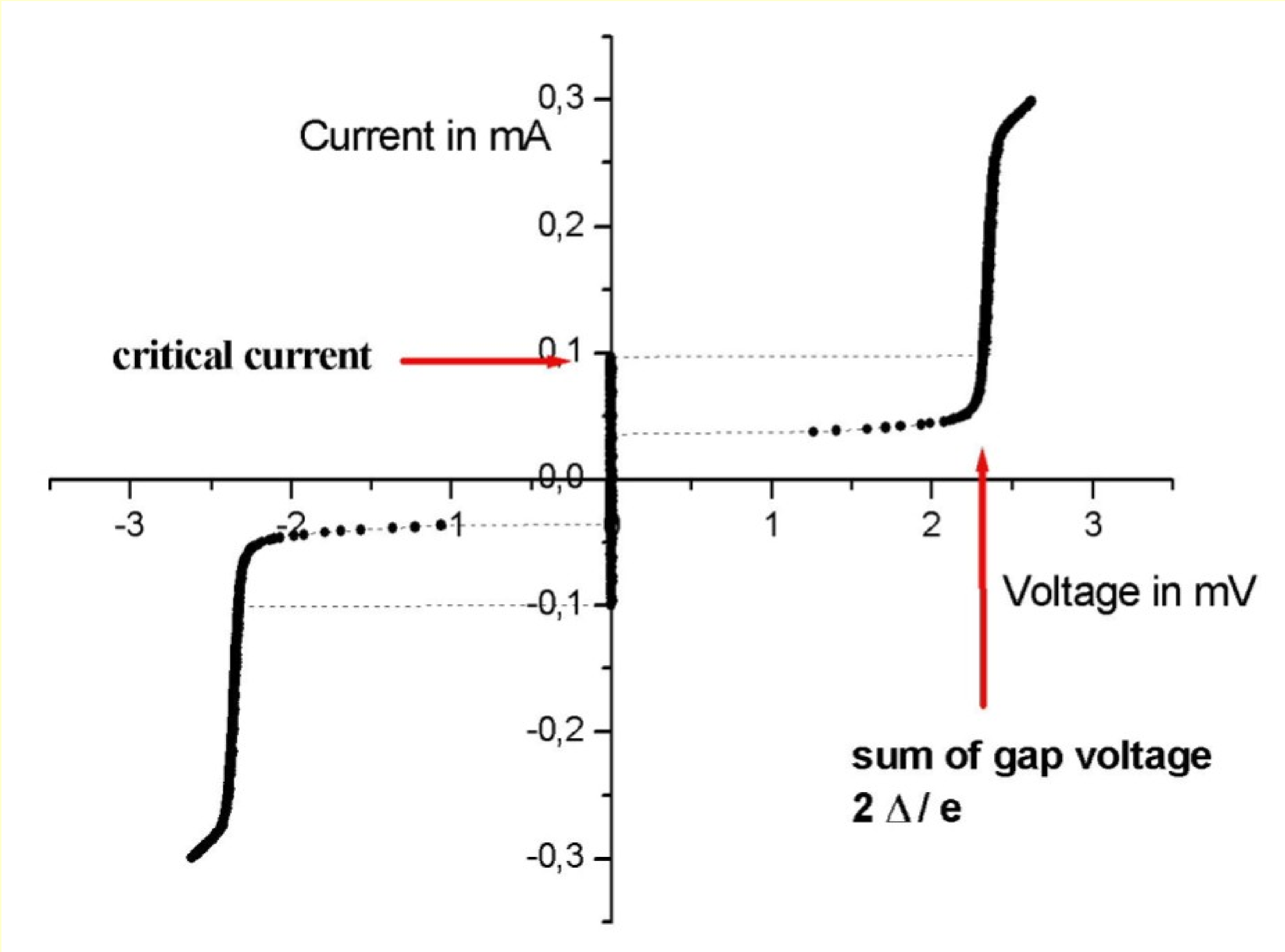
$$I_{qp_1} = \frac{\hbar}{\pi e R_N} \mathrm{P}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \frac{n_L(\omega) n_R(\omega')}{\omega - \omega' - \omega_0} [f(\omega) - f(\omega')]$$

$$I_{qp} = -\frac{\hbar}{e R_N} \int_{-\infty}^{+\infty} d\omega\, n_L(\omega) n_R(\omega - \omega_0) [f(\omega) - f(\omega - \omega_0)]$$

$$I_{J1} = -\frac{\hbar}{\pi e R_N} \mathrm{P}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \frac{p_L(\omega) p_R(\omega')}{\omega - \omega' - \omega_0} [f(\omega') - f(\omega)]$$

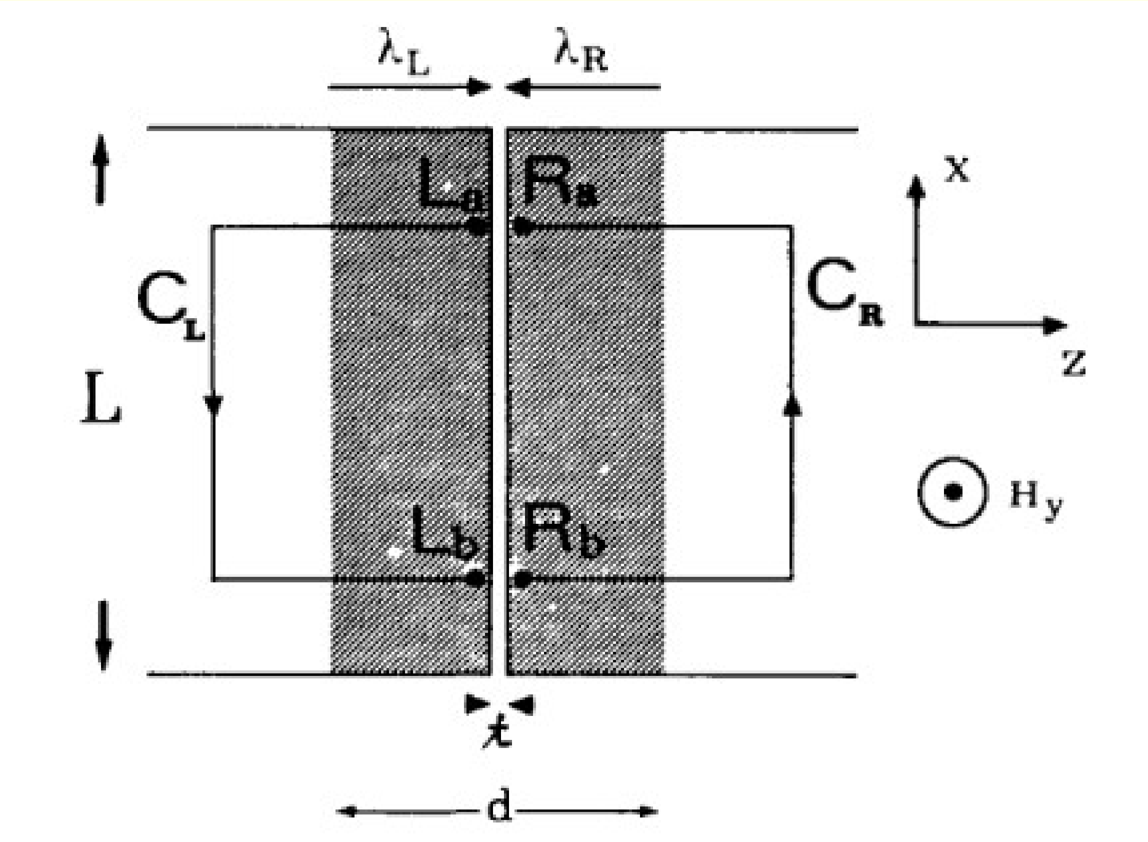
$$\omega_0 = \frac{\omega_f}{2} = \frac{eV_0}{\hbar}$$

$$I_{J2} = \frac{\hbar}{e R_N} \int_{-\infty}^{+\infty} d\omega\, p_L(\omega) p_R(\omega - \omega_0) [f(\omega - \omega_0) - f(\omega)]$$



Current in mA
0,3
0,2
0,1
0,0
-0,1
-0,2
-0,3
-3
-2
-1
0
1
2
3
critical current
Voltage in mV
sum of gap voltage
2 Δ / e

Magnetic field effects



$$\nabla \varphi_{L,R} = \frac{2e}{\hbar c}\left(\frac{mc}{2e^2\rho}\mathbf{J}_S + \mathbf{A}\right)$$

$$\nabla \times \mathbf{A} = \mathbf{H}.$$

$$\varphi(x+dx) - \varphi(x) = \frac{2e}{\hbar c}\oint \mathbf{A}\cdot\mathbf{dl}$$

$$\varphi_{Ra}(x) - \varphi_{Rb}(x+dx) = \frac{2e}{\hbar c}\int_{C_R}\left(\mathbf{A} + \frac{mc}{2e^2\rho}\mathbf{J}_S\right)\cdot\mathbf{dl}$$

$$\varphi_{Lb}(x+dx) - \varphi_{La}(x) = \frac{2e}{\hbar c}\int_{C_L}\left(\mathbf{A} + \frac{mc}{2e^2\rho}\mathbf{J}_S\right)\cdot\mathbf{dl}$$

$$\oint \mathbf{A}\cdot\mathbf{dl} = H_y(\lambda_L + \lambda_R + \mathrm{t})\,dx$$

$$\frac{d\varphi}{dx} = \frac{2e}{\hbar c}(\lambda_L + \lambda_R + \mathrm{t})H_y$$

$$\frac{d\varphi}{dx} = \frac{2e}{\hbar c}(\lambda_L + \lambda_R + t)H_y$$

$$\varphi = \frac{2e}{\hbar c} d H_y x + \varphi_0$$

$$d = (\lambda_L + \lambda_R + t)$$

$$J = J_1 \sin\left(\frac{2e}{\hbar c} d H_y x + \varphi_0\right)$$

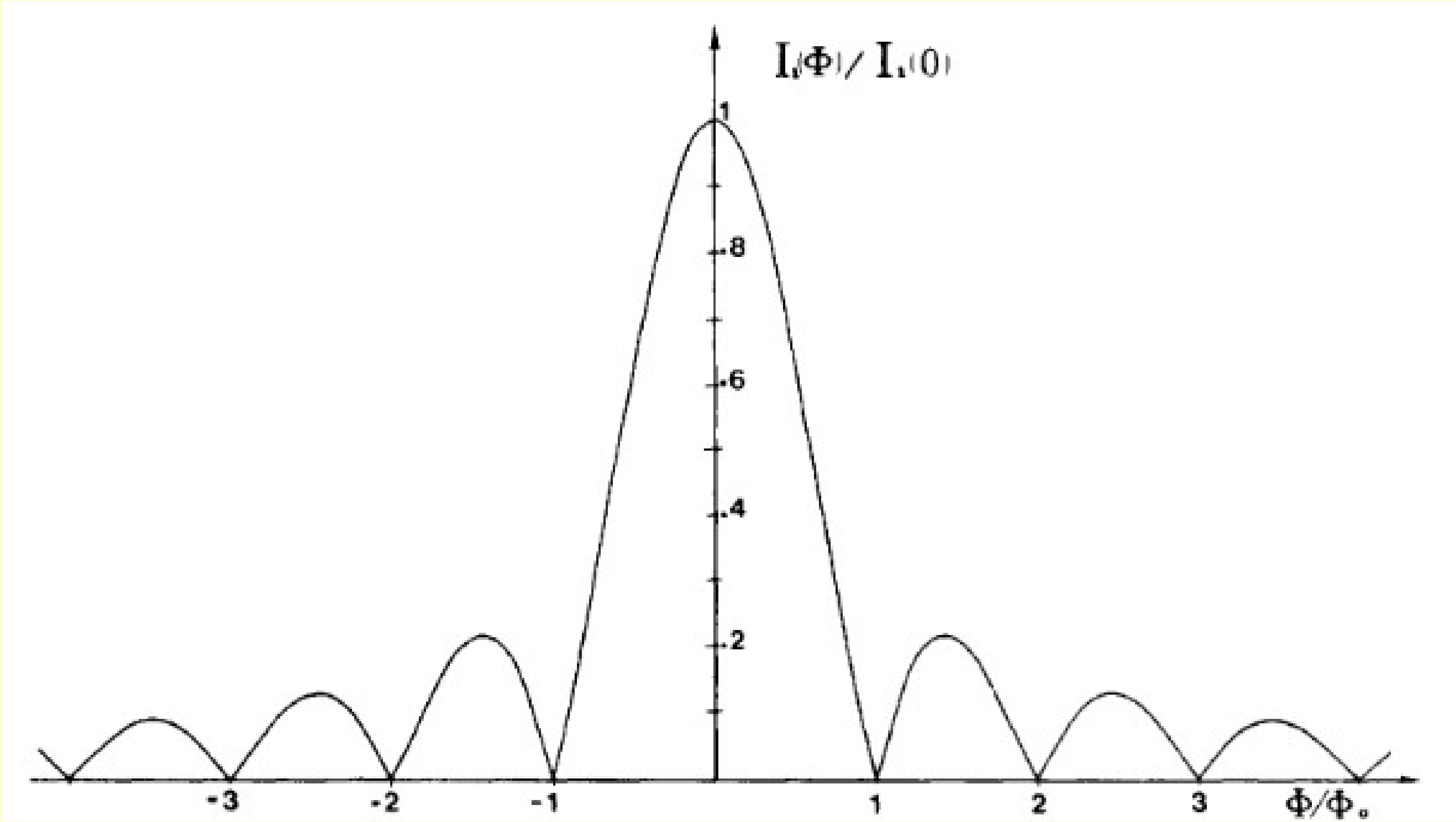
this indicates that the tunneling supercurrent is spatially modulated by the magnetic field. Then, due to the periodic character of the expression, situations can be realized in which the net tunneling current is zero.

In particular a rectangular junction with a uniform zero field tunneling current distribution exhibits a dependence of the maximum supercurrent on the applied magnetic field in the form of a Fraunhofer-like diffraction pattern.

$$I_1(H) = I_1(0)\left|\frac{\sin \pi \frac{\Phi}{\Phi_0}}{\pi \frac{\Phi}{\Phi_0}}\right|$$

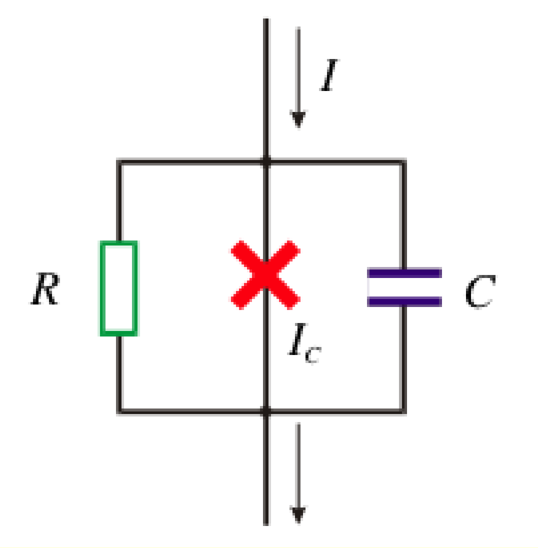
Φ is the total magnetic flux threading the junction

Φ_0 = hc/2e is the flux quantum = 2.07 10^{-7} G cm^2



Theoretical magnetic field dependence of the maximum Josephson current for a rectangular junction.

The resistively shunted model RSJ



$$I = I_c \sin\varphi + \frac{V}{R} + C\frac{dV}{dt}$$

$$\frac{d\varphi}{dt} \equiv \dot{\varphi} = \frac{2\pi}{\Phi_0} V$$

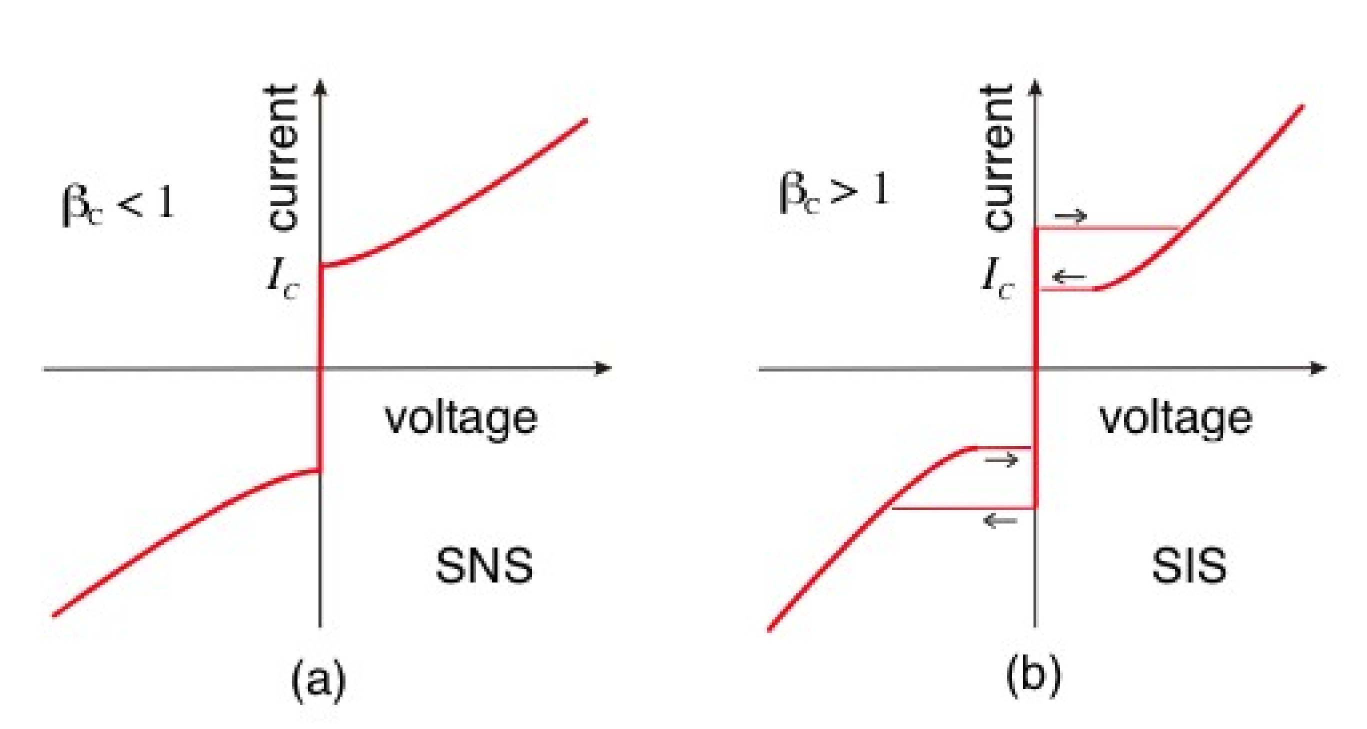
$$I = I_c \sin\varphi + \frac{\Phi_0}{2\pi R}\dot{\varphi} + \frac{\Phi_0 C}{2\pi}\ddot{\varphi}$$

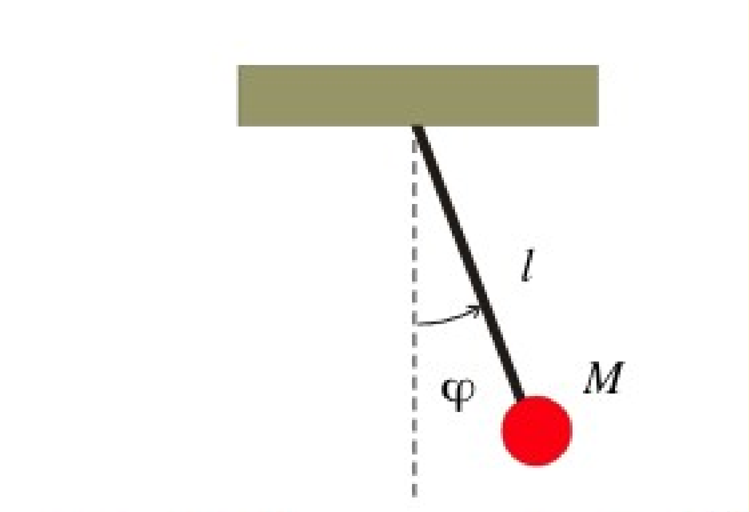
$$i \equiv \frac{I}{I_c}, \quad \tau = \frac{2\pi I_c R}{\Phi_0} t$$

$$\beta_c \frac{d^2\varphi}{d\tau^2} + \frac{d\varphi}{d\tau} + \sin\varphi = i$$

$$\beta_c = \frac{2\pi I_c R^2 C}{\Phi_0}$$

McCumber parameter

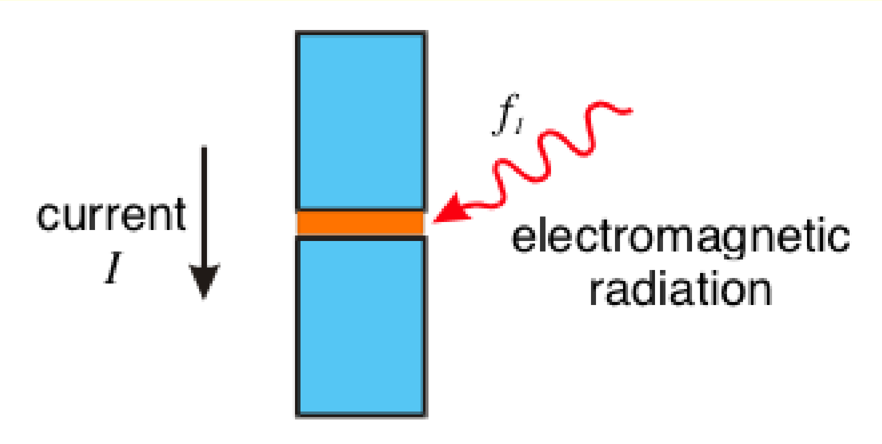




$$T = Mgl\sin\varphi + \chi\dot{\varphi} + \Theta\ddot{\varphi}$$

RF effects: Shapiro steps

$$V = V_0 + V_1 \cos(2\pi f_1 t)$$



$$\varphi = \int \frac{2\pi}{\Phi_0} V\,dt = \varphi_0 + \frac{2\pi}{\Phi_0} V_0 t + \frac{V_1}{\Phi_0 f_1}\cos(2\pi f_1 t)$$

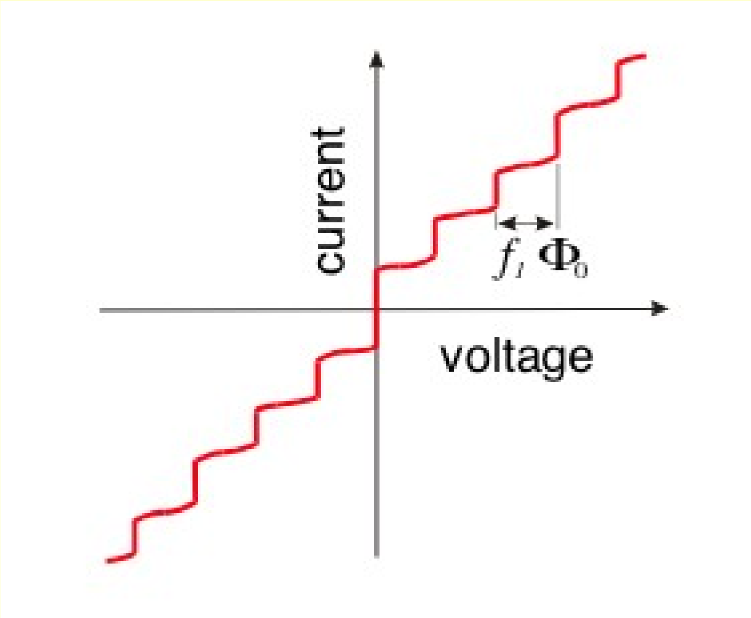
$$\sin(z \sin x) = 2 \sum_{k=0}^{\infty} J_{2k+1}(z)\,\sin(2k+1)x$$

$$\cos(z \sin x) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z)\cos 2kx$$

$$I_s = I_c \sin\varphi = I_c \sin\left[\varphi_0 + \frac{2\pi}{\Phi_0} V_0 t + \frac{V_1}{\Phi_0 f_1}\cos(2\pi f_1 t)\right]$$
$$= I_c \sin\left[\varphi_0 + \frac{2\pi}{\Phi_0} V_0 t\right]\cos\left[\frac{V_1}{\Phi_0 f_1}\cos(2\pi f_1 t)\right]$$
$$+ I_c \cos\left[\varphi_0 + \frac{2\pi}{\Phi_0} V_0 t\right]\sin\left[\frac{V_1}{\Phi_0 f_1}\cos(2\pi f_1 t)\right]$$

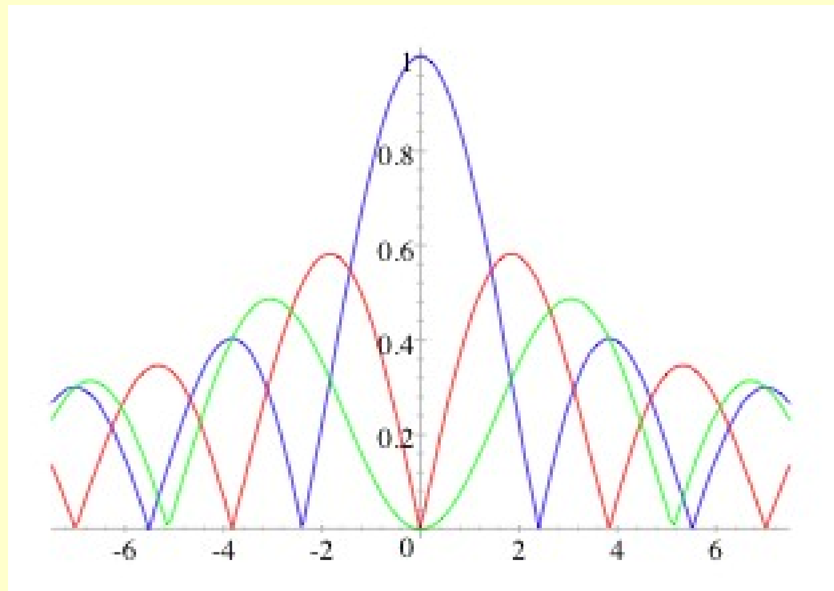
$$I_s = I_c \sum_{n=0}^{\infty} (-1)^n J_n\left(\frac{V_1}{\Phi_0 f_1}\right)\sin\left[\varphi_0 + \frac{2\pi}{\Phi_0} V_0 t - 2\pi n f_1 t\right]$$

At specific voltages the current is dc $\frac{2\pi}{\Phi_0} V_0 = 2\pi n f_1$ $V_0 = n f_1 \Phi_0,\ n = 0, \pm 1, \pm 2, \ldots$



The amplitude of the constant voltage steps follows the Bessel function profile and depends on the RF field amplitude

$$\Delta I_n \simeq I_c J_n\left(\frac{V_1}{\Phi_0 f_1}\right)$$



Electrodynamics of a Josephson junction

Assuming non zero H in x and y

$$\frac{\partial \varphi}{\partial x} = \frac{2e}{\hbar c} H_y d$$

$$\frac{\partial \varphi}{\partial y} = -\frac{2e}{\hbar c} H_x d$$

and using Maxwell equation

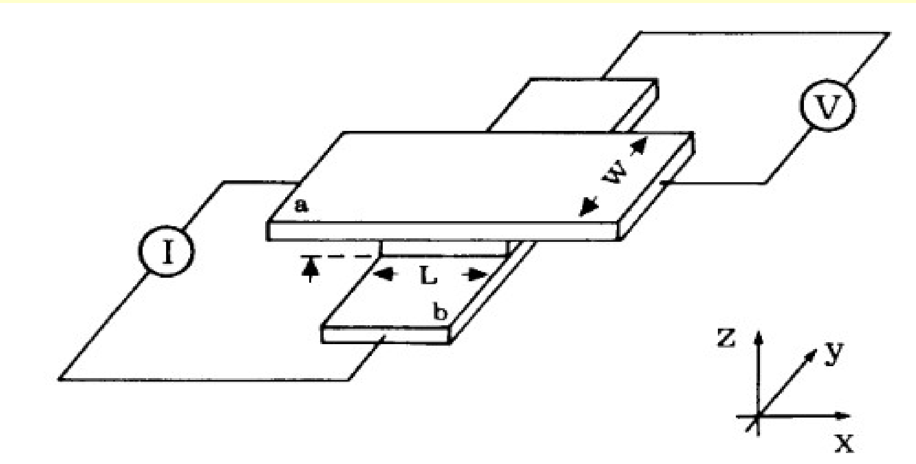
$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{4\pi}{c} J_z + \frac{1}{c}\frac{\partial D_z}{\partial t}$$

$$\frac{\hbar c^2}{8\pi e d}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = J_1 \sin\varphi + C\frac{dV}{dt}$$

$$C = \varepsilon_r / 4\pi t$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \frac{1}{\bar{c}^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\lambda_J^2}\sin\varphi$$



$$\lambda_J = \left(\frac{\hbar c^2}{8\pi e d J_1}\right)^{1/2}$$

$$\bar{c} = c\left(\frac{1}{4\pi C d}\right)^{1/2} = c\left(\frac{t}{\varepsilon_r d}\right)^{1/2}$$

From microscopic theory of Josephson effect

$$J = J_1(V)\sin\varphi + [\sigma_1(V)\cos\varphi + \sigma_0(V)]V$$

$$J = J_1 \sin\varphi + \sigma_0(V)V$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\beta}{\bar{c}^2}\frac{\partial\varphi}{\partial t} = \frac{1}{\lambda_J^2}\sin\varphi$$

$$\beta = \sigma_0 / C.$$

Nonlinear wave equation (sine-Gordon equation)

In the case of spatially constant φ

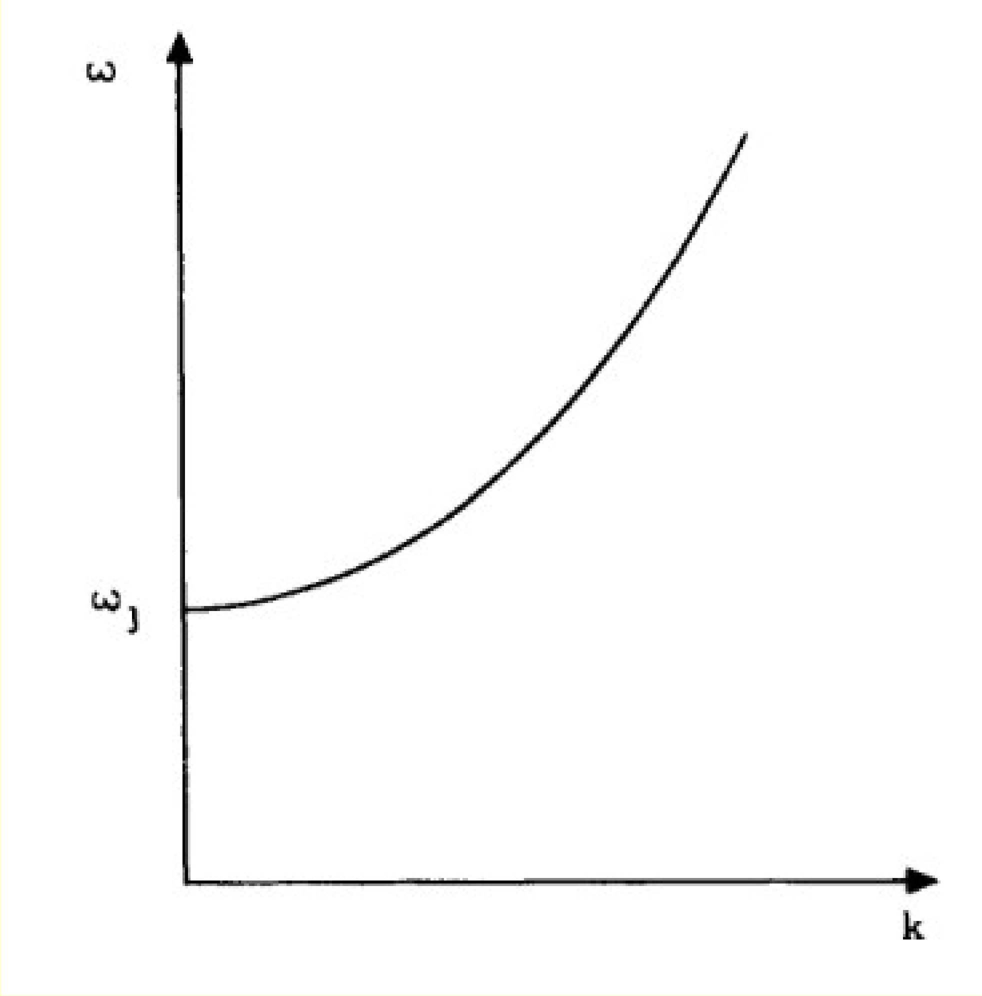
$$\frac{d^2\varphi}{dt^2} + \omega_J^2 \sin\varphi = 0$$

$$\omega_J = \bar{c}/\lambda_J.$$

Pendulum equation

For small amplitude travelling wave solutions
of the type $\varphi \sim \exp[\, j\,(\omega t - kx)\,]$
It is easy to derive the dispersion relation as

$$\omega^2 = \omega_J^2 + k^2 \bar{c}^2$$



Nonlinear waves in Josephson junctions

When the phase φ is spatially dependent
i.e. one of the junction dimensions is > λ_J

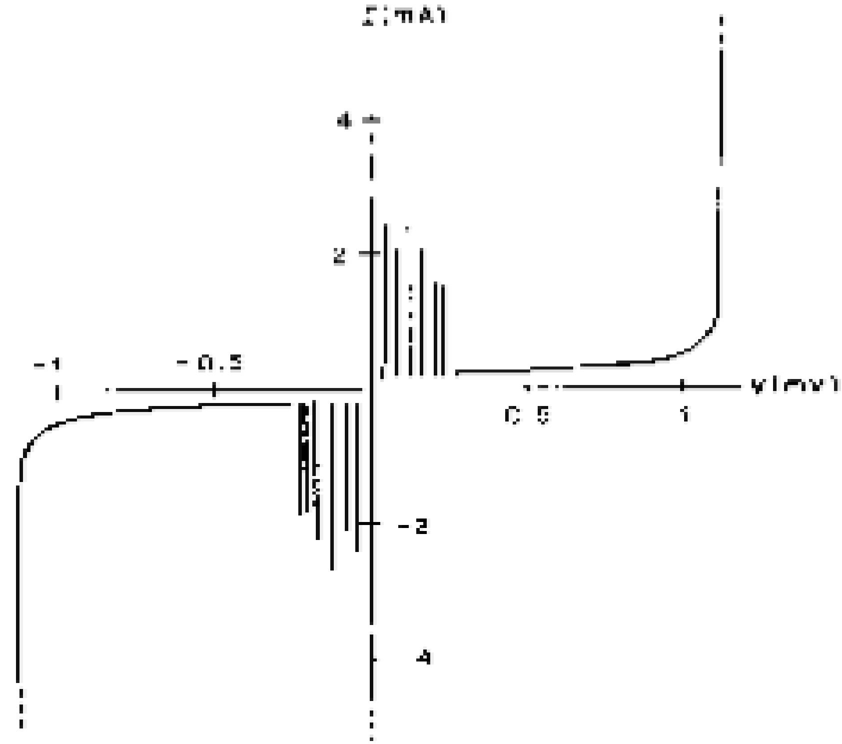
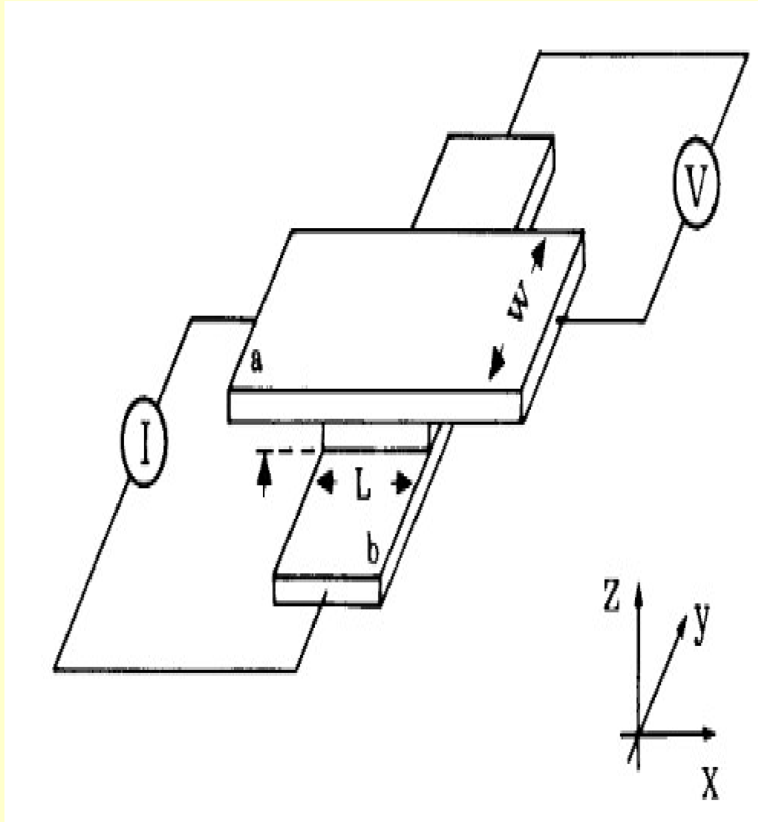


FIG. 2. The I (V) curve of a long, narrow tunnel junction showing the vortex-propagation branches. The near symmetry in ±V indicates that the stray and applied magnetic fields are nearly zero.



T.A.Fulton and R.C.Dynes, Solid St. Commun. **12**, 57 (1972)

In the case of a long and narrow Josephson junction the wave equation becomes:

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi + \alpha\varphi_t - \beta\varphi_{xxt} - \gamma$$

Where the following normalization has been done: $\frac{x}{\lambda_J} \to x \; ; \quad \frac{t}{\omega_p^{-1}} \to t$

$$\omega_p = \sqrt{\frac{2\pi j_c}{\Phi_0 C}} \qquad \lambda_J = \sqrt{\frac{\Phi_0}{2\pi j_c \mu_0 (2\lambda_L + t_0)}} \qquad \bar{c} = \lambda_J \omega_p = \frac{1}{\sqrt{L'C}} = c_0 \sqrt{\frac{t_0}{\varepsilon(2\lambda_L + t_0)}}$$

Plasma frequency - Josephnon penetration length - Swihart velocity

$$\varphi_{xx} \equiv \frac{\partial^2\varphi}{\partial x^2}, \; \varphi_{tt} \equiv \frac{\partial^2\varphi}{\partial t^2}, \; \varphi_t \equiv \frac{\partial\varphi}{\partial t}.$$

$\alpha = \frac{1}{RC\omega_p}$ Normalized damping parameter = $\beta_c^{-1/2}$

$\beta = \frac{\omega_p L'}{R_s}$ Normalized surface impedance damping parameter

$\gamma = \frac{j_B}{j_c}$ Normalized bias current

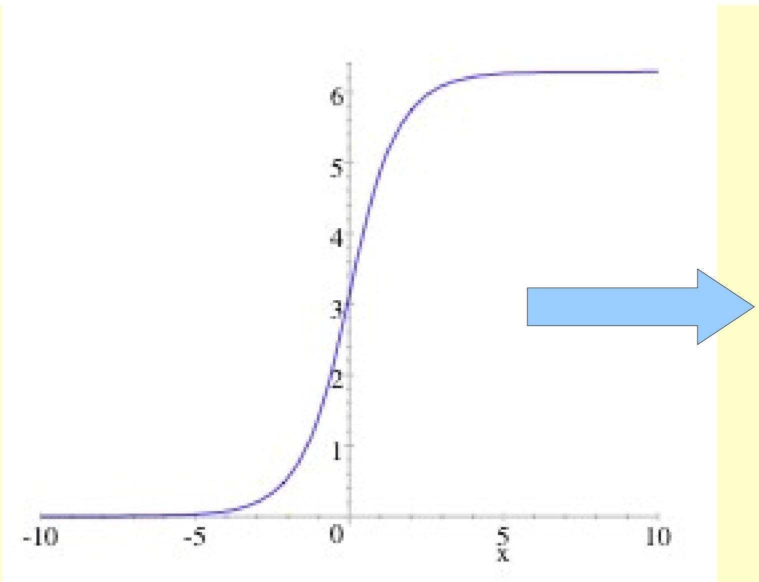
Normally α,β <<1 and γ < 1

If α=β=γ = 0, the equation becomes the sine-Gordon equation

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi$$

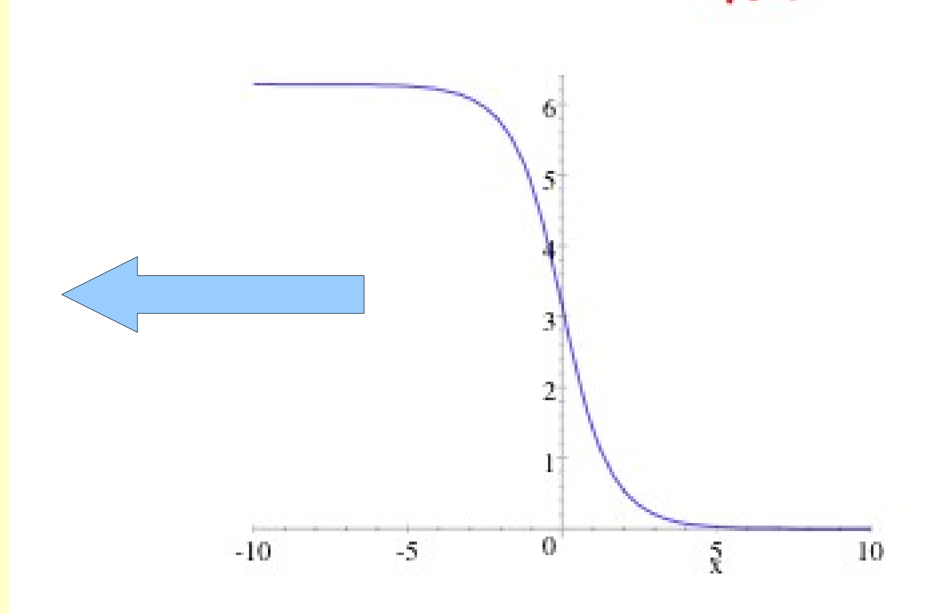
A solution is the kink travelling wave

$$\varphi_0(x,t) = 4\arctan[\exp\frac{x - vt - x_0}{\sqrt{1-v^2}}]$$



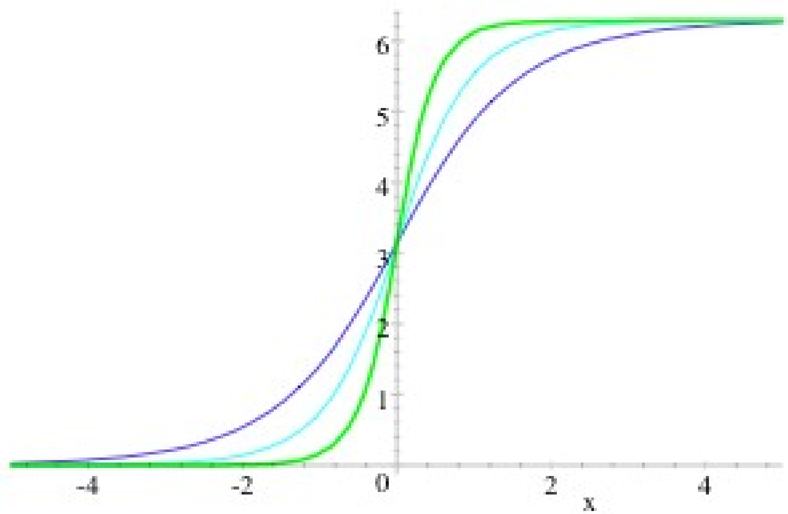
Also the anti-kink travelling wave

$$\overline{\varphi_0}(x,t) = 4\arctan[\exp(-\frac{x-vt-x_0}{\sqrt{1-v^2}})]$$

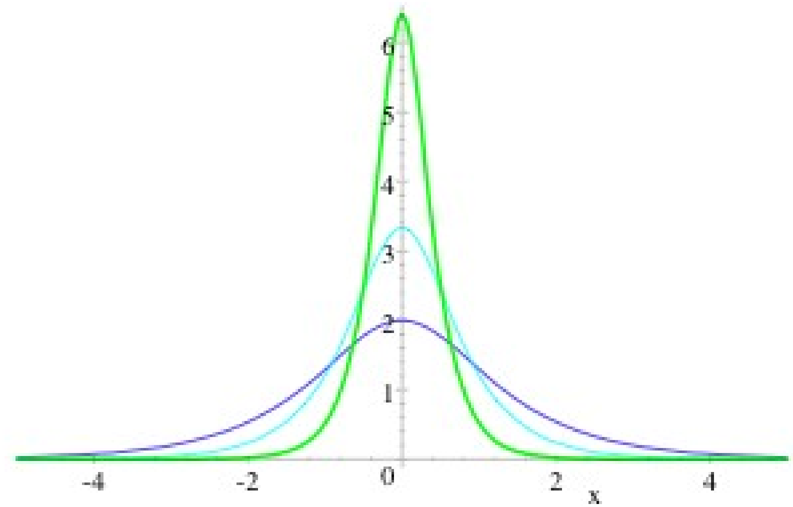


The sine-Gordon equation is invariant under Lorentz transformations

$$x \to x' = \frac{x - vt}{\sqrt{1 - v^2}}, \quad t \to t' = \frac{t - x/v}{\sqrt{1 - v^2}}$$



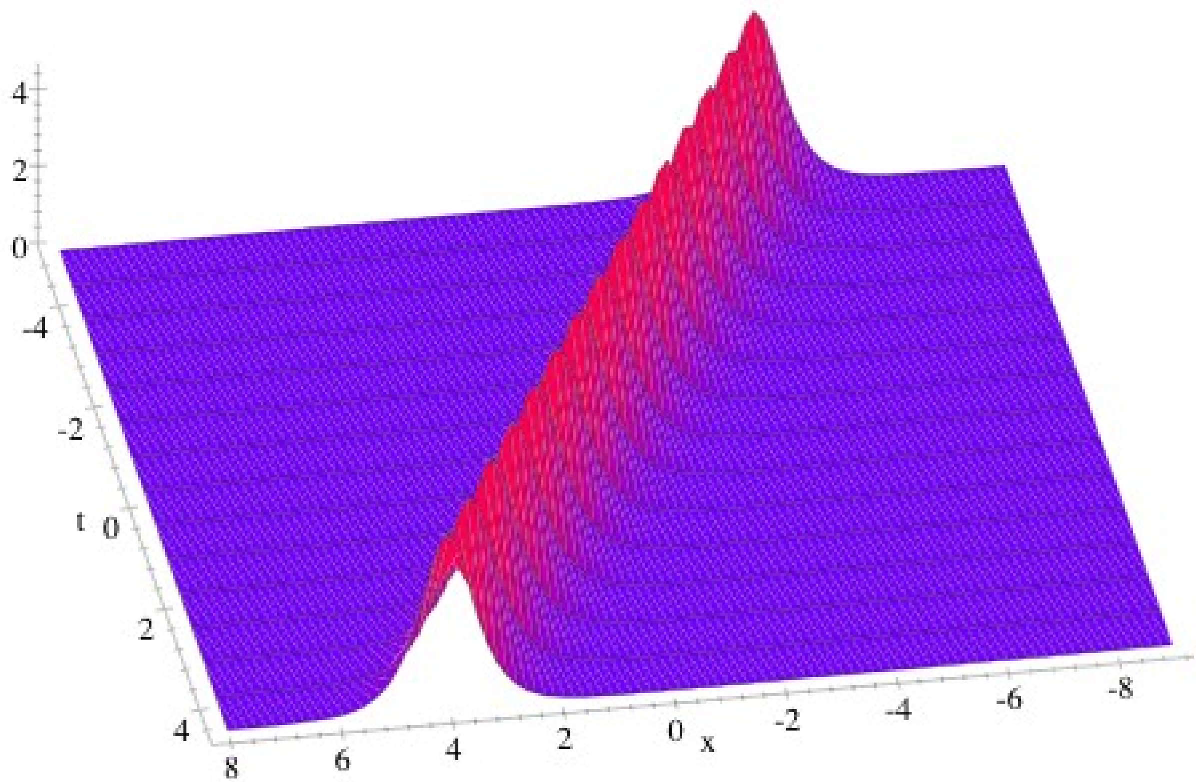
Lorentz contraction of $\overline{\varphi_0}(x,0)$: $v = 0$ (blue), $v = 0.8$ (cyan), $v = 0.95$ (green).



Lorentz contraction of $\frac{d}{dx}[\varphi_0(x,0)]$:

as dφ/dx is proportional to H this represents the local magnetic field (a flux quanta or fluxon)

The fluxon behaves as a relativistic particle with rest mass = 8



A soliton $\frac{d\varphi_0}{dx}$ moving with $v = 0.9$

$$m_0 = 8,\ m(v) = \frac{m_0}{\sqrt{1-v^2}}$$

$$\int_{-\infty}^{+\infty} \frac{d}{dx}[\varphi_0(x,0)]dx = 2\pi$$

The area under a fluxon is quantized to flux quantum Φ_0

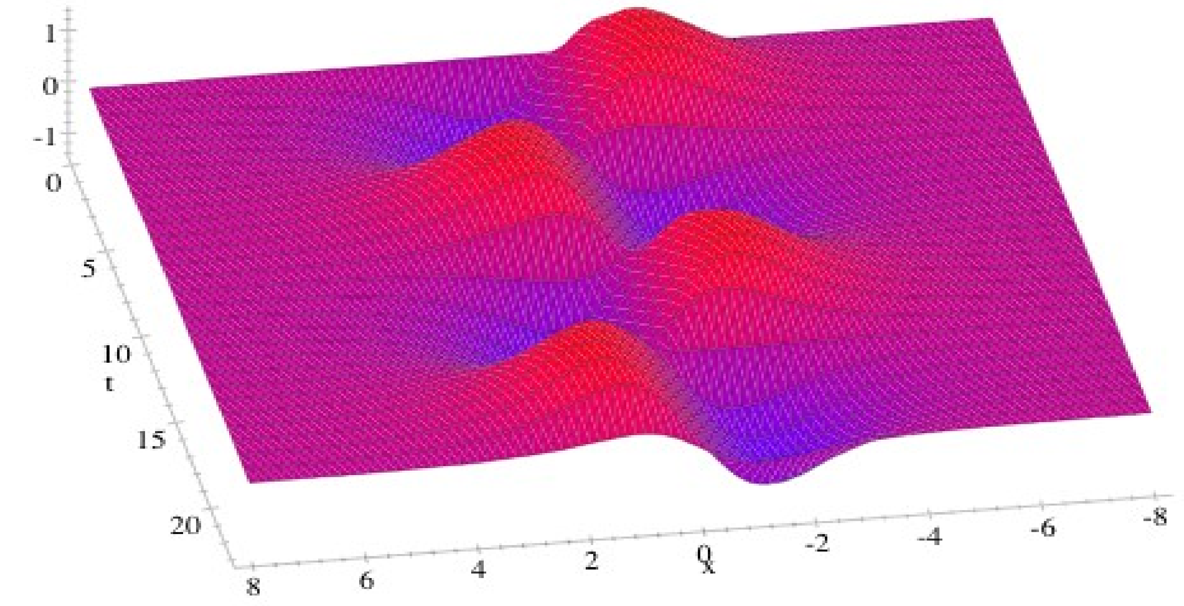
Other solutions of SGE are:

$$\varphi = 2\arcsin[k\,\mathrm{sn}(\frac{x-vt-x_0}{\sqrt{v^2-1}};k)]$$

Plasma waves

$$\varphi_{br}(x,t) = 4\arctan[\tan\theta\frac{\sin(t\cos\theta)}{\cosh(x\sin\theta)}]$$

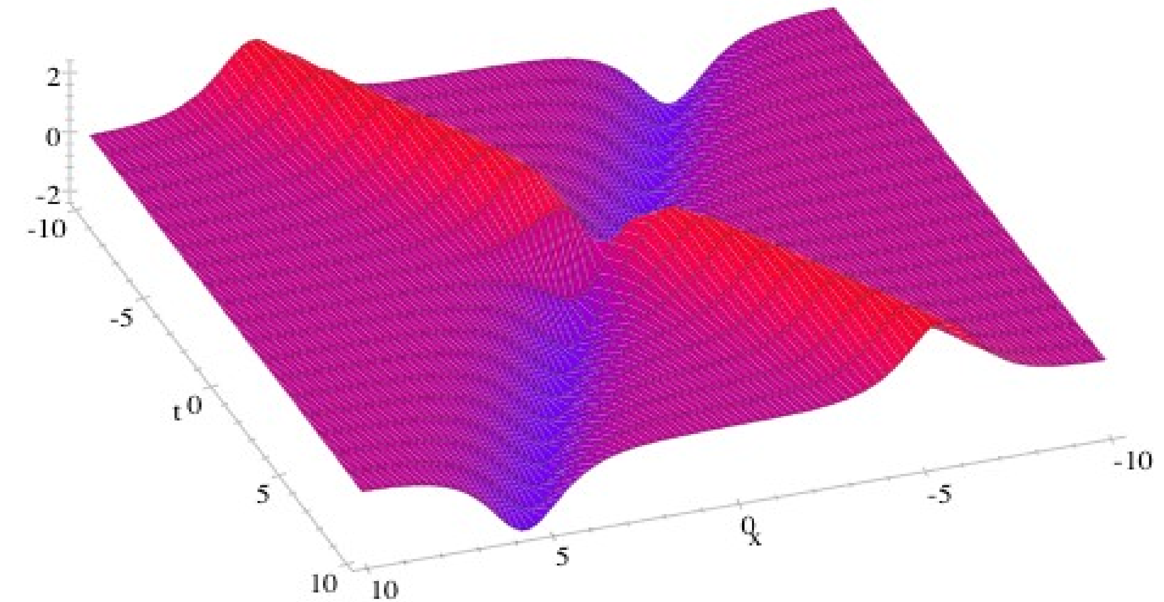
Breather



A breather with $\theta = 1$ and $v = 0$

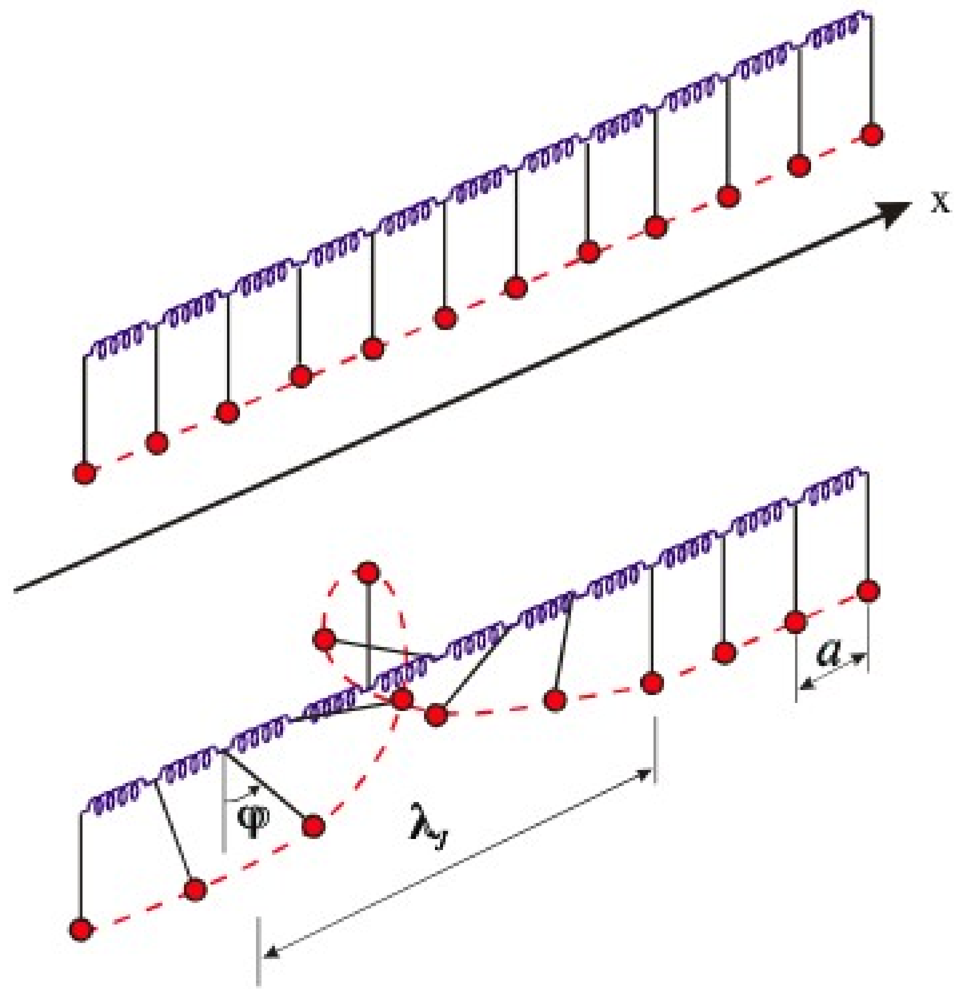
$$\varphi_{s\bar{s}} = 4\arctan[\frac{1}{v}\frac{\sinh(\frac{vt}{\sqrt{1-v^2}})}{\cosh(\frac{x}{\sqrt{1-v^2}})}]$$

Kink-anti-kink collision

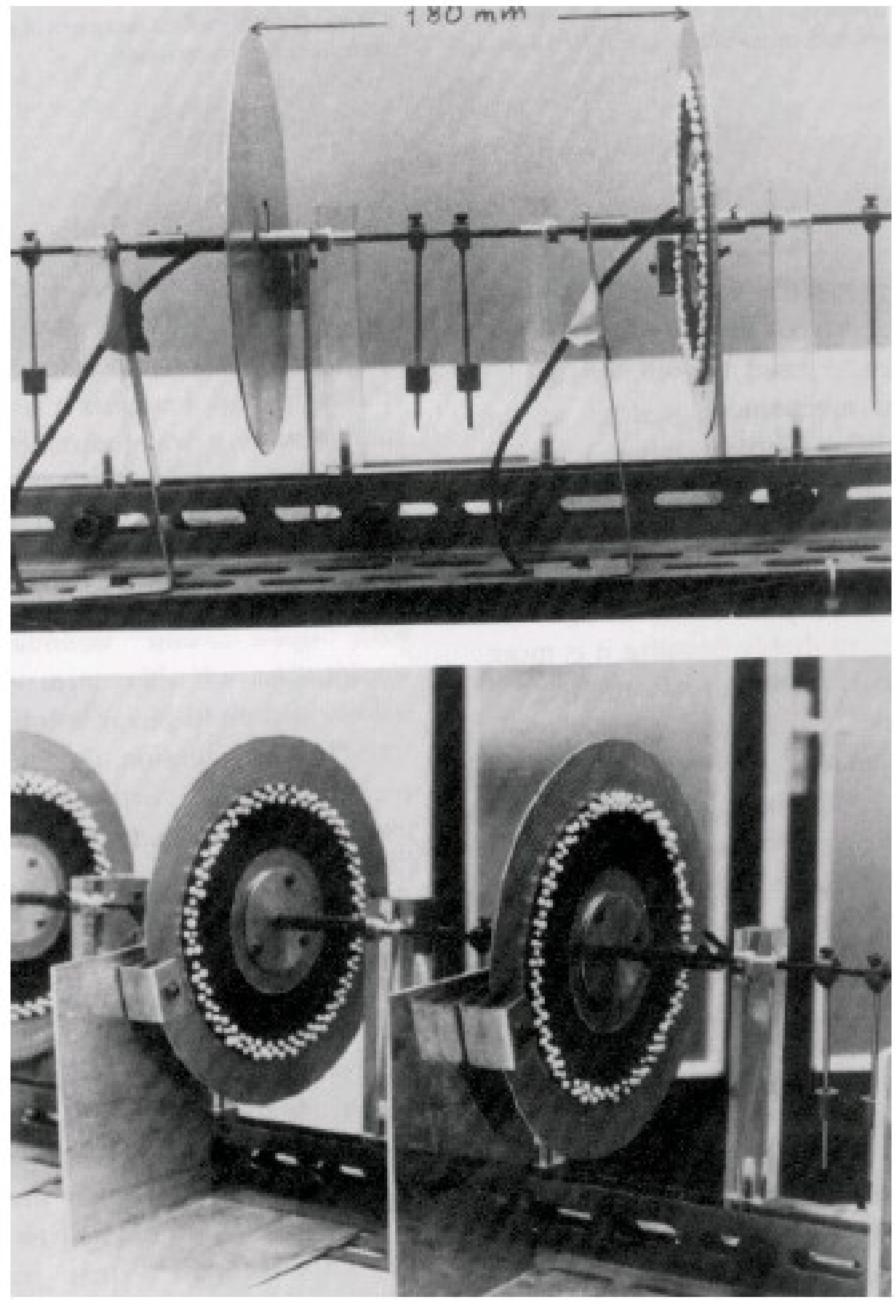


Soliton-antisoliton collision with $v = 0.5$

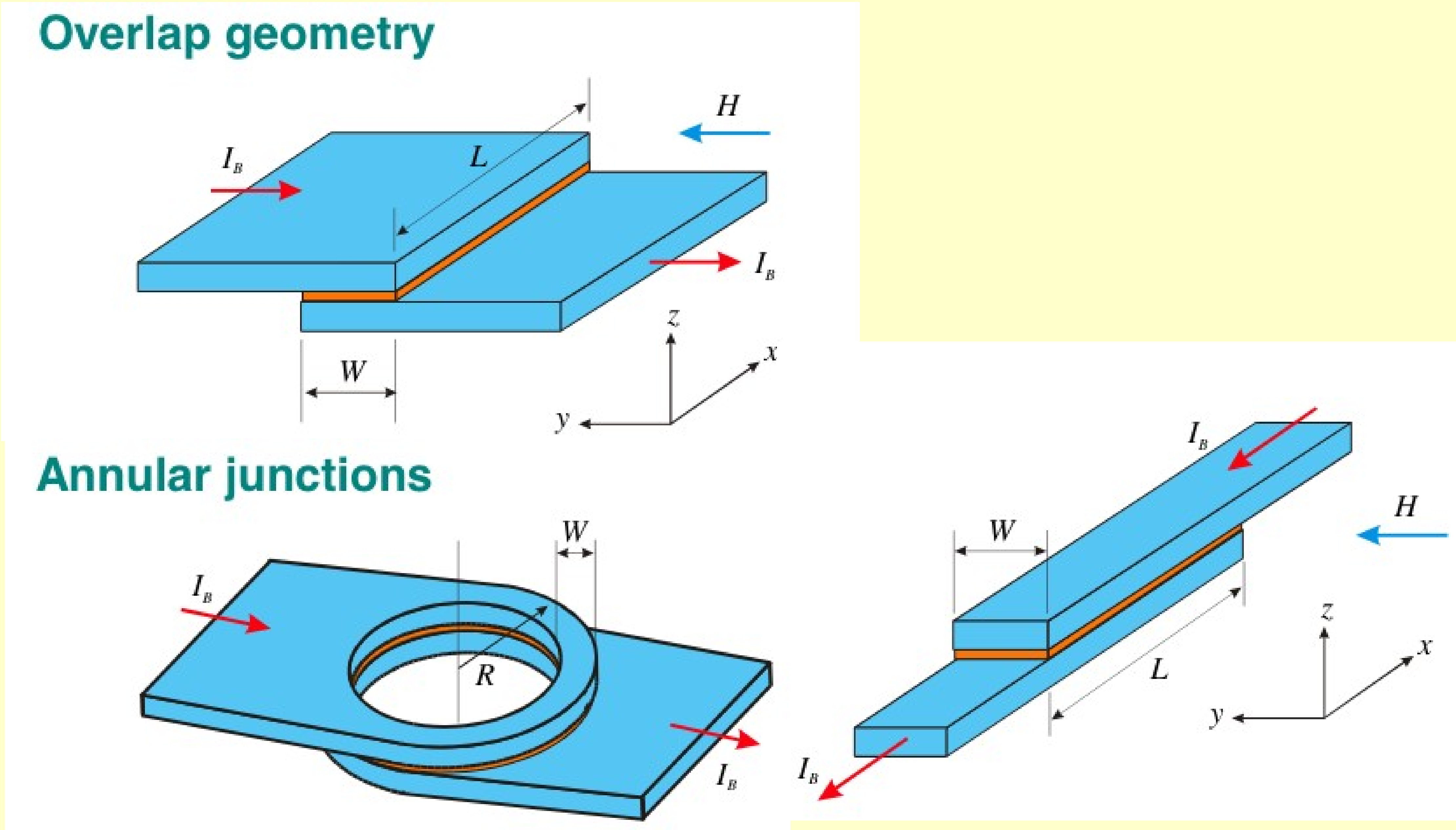
Mechanical analog of sine-Gordon equation



Detailed experimental study ($N = 25$):
M.Cirillo, R.D.Parmentier, and B.Savo,
Physica D **3**, 565 (1981).



Types of juntion geometries



Real case: α,β, and γ > 0 the perturbed sine-Gordon equation

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi + \alpha\varphi_t - \beta\varphi_{xxt} - \gamma$$

For the sine-Gordon equation we can write an Hamiltonian functional such that $dH^{SG}/dt = 0$ for each solution.

$$H^{SG} \equiv \int_{-\infty}^{\infty} \left(\tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_t^2 + 1 - \cos\varphi\right)dx$$

Using a kink solution in the case of the perturbed SGE we have :

$$\frac{dH^{SG}(\varphi)}{dt} = \int_{-\infty}^{\infty} \left(-\alpha\varphi_t^2 - \beta\varphi_{xt}^2 + \gamma\varphi_t\right)dx$$

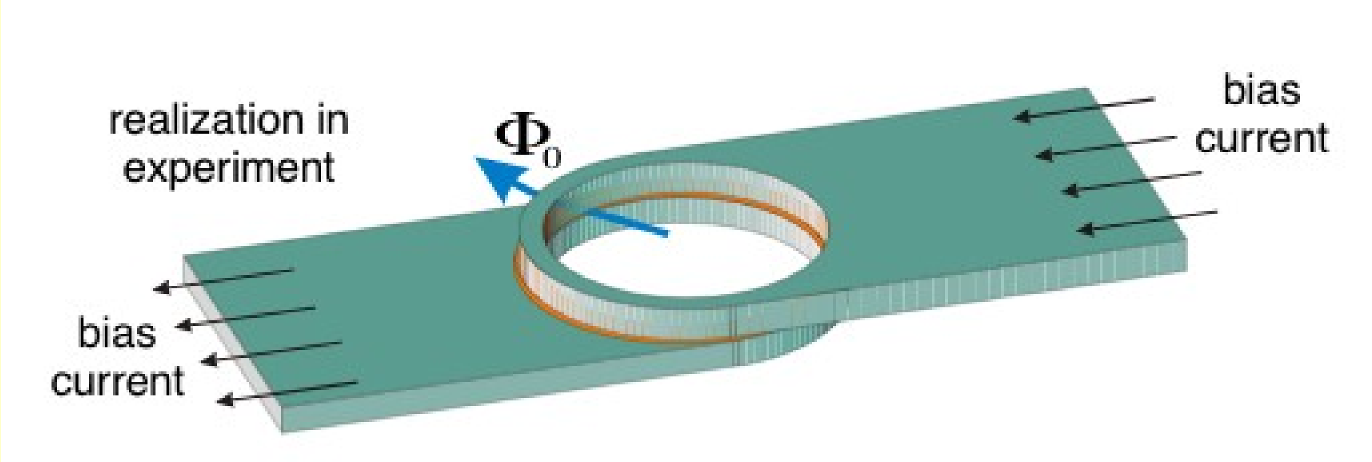
Assuming that the effect of the perturbations is just a velocity modulation of the kink

$$\frac{dv}{dt} = -\alpha v(1 - v^2) - \tfrac{1}{3}\beta v - \tfrac{1}{4}\pi\gamma(1 - v^2)^{3/2}$$

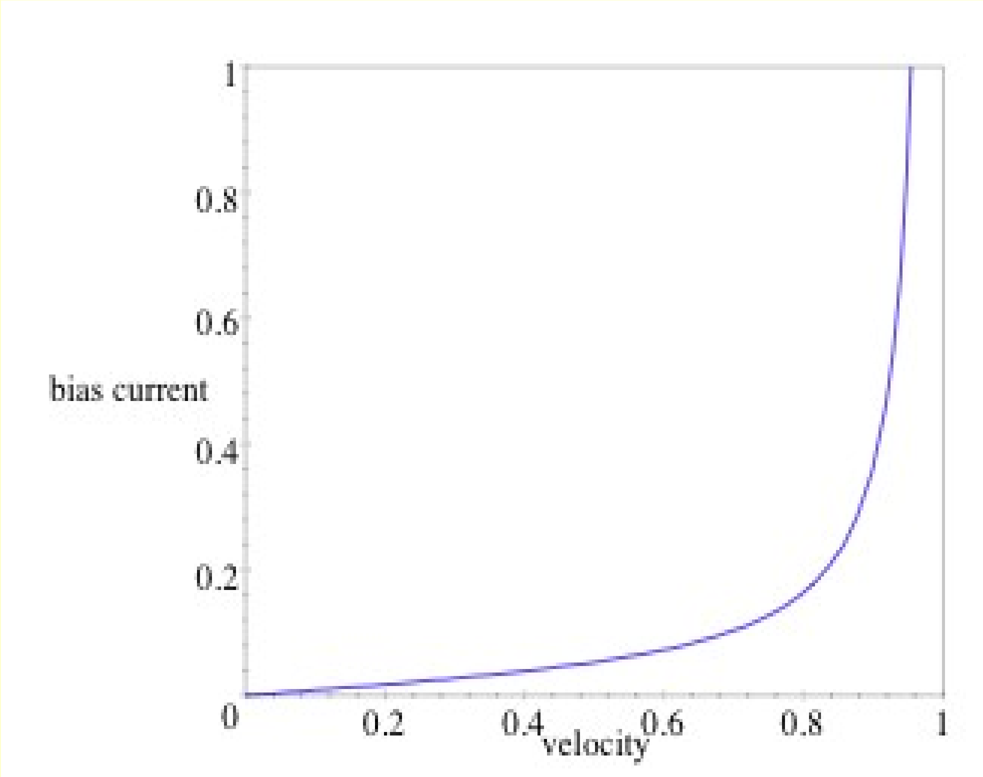
At equilibrium a constant velocity v_∞ is obtained

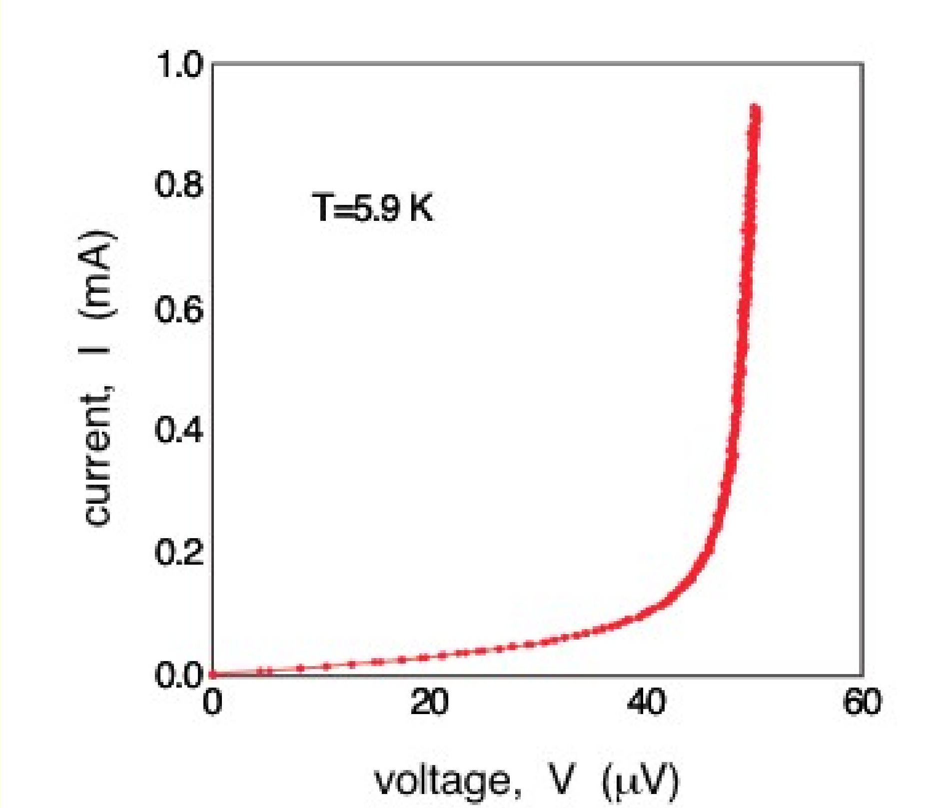
$$\gamma = \frac{4|v_\infty|}{\pi\sqrt{1 - v_\infty^2}}\left(\alpha + \frac{\beta}{3(1 - v_\infty^2)}\right)$$

Current-Velocity (Voltage) characteristics of a single fluxon



Annular junction with $n = 1$ fluxon





Experimental $\gamma(v_\infty)$ of a fluxon

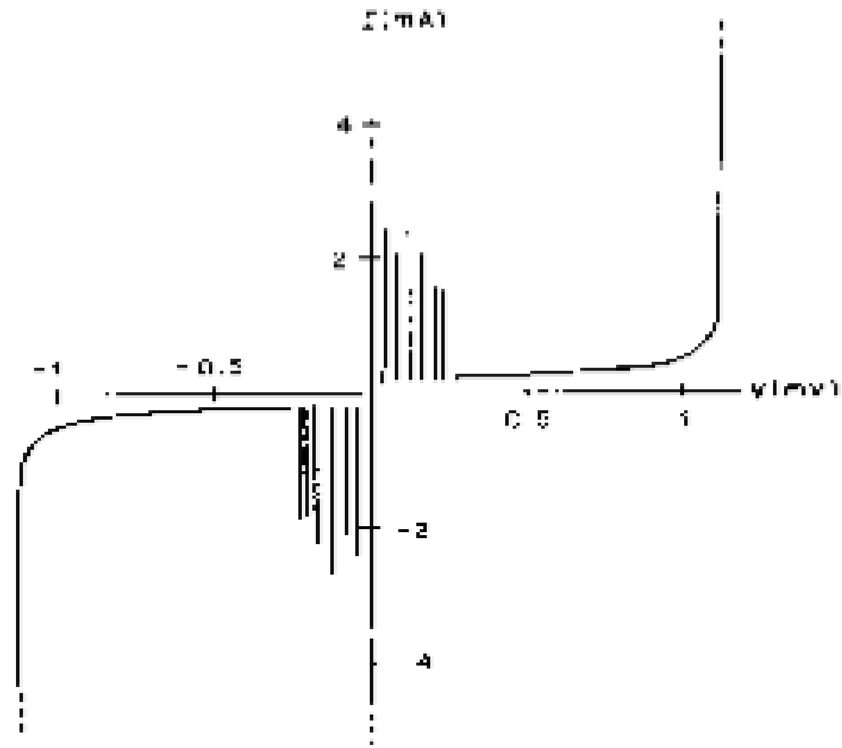
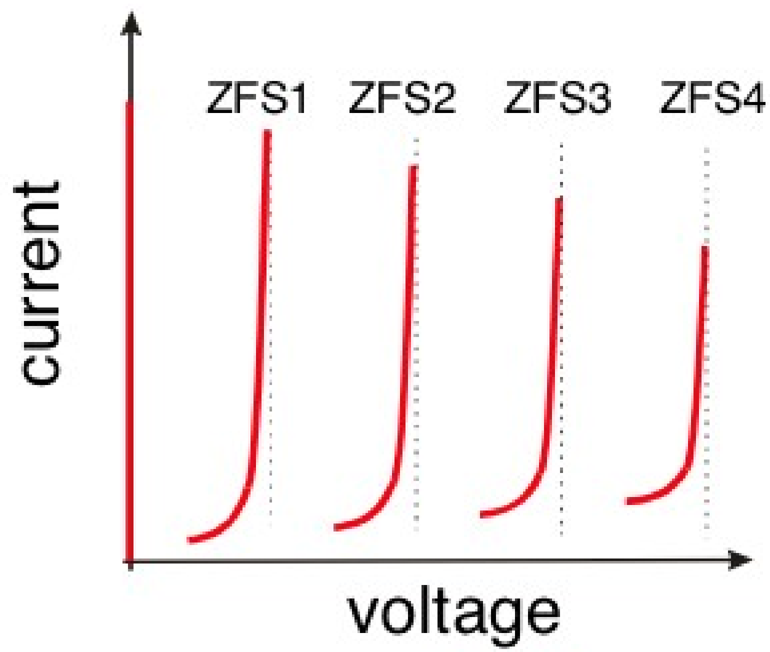


FIG. 2. The I – (V) curve of a long, narrow tunnel junction showing the vortex-propagation branches. The near symmetry in ΔI indicates that the stray and applied magnetic fields are nearly zero.



Overlap junction: $I - V$ curve

Concluding remarks

Useful references

Physics and Applications of the Josephson Effect by A. Barone and G. Paternò,1982 by John Wiley & Sons, Inc., NY

A.V. Ustinov, Lecture notes